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# Wave forces on steeply-sloping sea walls

P.J. KACHOYAN and W.D. MCKEE

*School of Mathematics, University of New South Wales, Box 1, Kensington, N.S.W. 2033, Australia*

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**Summary**

The forces exerted by non-breaking, normally-incident water waves on a sloping sea wall are investigated within the framework of linearised potential theory. The slope of the sea wall is assumed to be large. The solution is in the form of an eigenfunction expansion, the coefficients of which are found by two methods. The first is a perturbation scheme based on the smallness of the reciprocal of the slope and is carried out to second order in this quantity. The second is a Galerkin technique. Results are presented for the case of a planar, outward-sloping sea wall. In shallow water it is found that the normal wave force decreases as the slope of the wall increases. In deep water, the reverse is true whilst in water of intermediate depth the normal wave force is only weakly dependent upon the slope of the sea wall.

## 1. Introduction

The theoretical study of water-wave problems began well over a century ago and it is indeed remarkable that there are still many unsolved problems, even within the framework of linearised potential theory. In recent years, much attention has been paid to the effects of changes in bottom topography. The interested reader is referred to Tuck [8] for a discussion of some of the classical problems which involve variations in depth. The great power of modern computers has meant that many new problems have been attacked and many new techniques brought into use. Good recent examples are provided by the work of Kirby and Dalrymple [3] who studied propagation across a trench and the use of the "parabolic approximation" by Radder [6] and others.

The problem to be addressed in this paper is that of the reflection of a normally-incident monochromatic wave train by a sloping sea wall. Of particular interest to engineers is the wave force exerted on the wall. The corresponding problem for an axisymmetric body has been treated by Fenton [2] who used a Green-function technique. Experimental measurements of the wave forces on walls have been carried out in the laboratory by Rundgren [7] who studied only vertical walls. We will confine ourselves to the case of waves for which the slope of the sea surface is always small and will use linearised potential theory. There are several techniques that could be used to attack this problem. For example, one could use a Green-function method or a source-distribution method. These are both well-known in the case of depth variations (see, for example, Macaskill [5]) and lead to integral equations which then have to be solved numerically.

Here, we will assume that the slope of the sea wall is large. This is likely to be true in many practical cases and means that we can set up a perturbation scheme in which the small parameter is a measure of the reciprocal of the slope of the sea wall. This leads to an eigenfunction expansion for the velocity potential. We calculate the coefficients in this expansion, and hence the force on the wall, correct to second order in the small parameter. An alternative approach is also used, namely to numerically determine the coefficients in the eigenfunction expansion by means of a Galerkin technique. In principle, this is not restricted to walls of great slope. However, it is found not to be numerically reliable if the slope is too "small" in a sense which will become more apparent later. Numerical results are presented and discussed for the case of a planar outward-sloping sea wall. It is found that the perturbation and Galerkin methods give results which merge as the wall slope increases and that the behaviour of the force as a function of slope is quite different in deep water as compared with shallow water. Both the perturbation and Galerkin methods can be readily extended to the case of obliquely-incident waves.

## 2. Formulation

We consider water waves in the region bounded by a flat bottom at $y = -d$, a free surface at $y = 0$ and a sea wall of profile $x = f(y)$, where $f(0) = 0$, for $-d < y < 0$. The geometry is shown in Figure 1. A wave of amplitude $a/2$ is assumed to be normally incident from $x = +\infty$. Since there is no mechanism for wave breaking or energy dissipation in the model, this wave will be totally reflected from the wall with amplitude $a/2$ and some phase change which will clearly depend upon the shape of the wall as well as the frequency of the incoming wave and the offshore depth.

Thus for $x \to +\infty$, we expect that the solution will behave like

$$\eta \sim a \cos(kx + \beta) \sin \omega t, \tag{1}$$

$$\phi \sim \frac{ga}{\omega} \frac{\cosh k(y+d)}{\cosh kd} \cos(kx + \beta) \cos \omega t, \tag{2}$$

where $\phi$ is the velocity potential, $\eta$ the free-surface elevation, $\omega$ the frequency and $k$ the wavenumber which satisfies the dispersion relation

$$\omega^2 = gk \tanh kd.$$

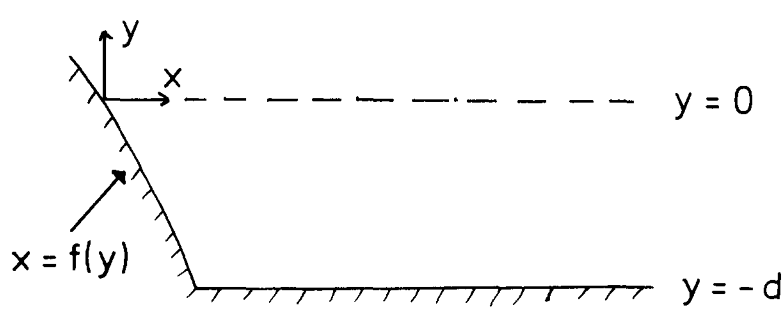


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Figure 1. The sea wall and the coordinate system used.

If the wall is vertical, $f(y) \equiv 0$ and (1), (2) form an exact solution of the linearised problem with $\beta = 0$.

We now define dimensionless variables, temporarily denoted by asterisks, as follows:

$$(x, y, \eta, f) = d(x^*, y^*, \eta^*, f^*), \ \omega = (g/d)^{1/2}\omega^*, \ t = (d/g)^{1/2}t^*,$$

$$\phi = (ga/\omega)\phi^*, \ p = \rho gap^*, \ k = k^*/d, \ \boldsymbol{F} = \rho gad\boldsymbol{F}^*$$

where $\rho$ is the water density, $p$ the pressure and $\boldsymbol{F}$ the wave force per unit span on the wall. When the asterisks are dropped, the problem becomes, in dimensionless variables,

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \qquad \text{for } -1 < y < 0 \text{ and } f(y) < x < \infty, \tag{3}$$

$$\frac{\partial\Phi}{\partial y} = 0 \qquad \text{at } y = -1 \quad \text{for } x > f(-1), \tag{4}$$

$$\frac{\partial\Phi}{\partial y} = \omega^2\Phi \qquad \text{at } y = 0 \quad \text{for } x > 0, \tag{5}$$

$$\frac{\partial\Phi}{\partial x} - f'(y)\frac{\partial\Phi}{\partial y} = 0 \qquad \text{on } x = f(y) \quad \text{for } -1 < y < 0, \tag{6}$$

and

$$\Phi \sim \operatorname{sech} k \cosh k(1 + y) \cos(kx + \beta) \qquad \text{as } x \to \infty, \tag{7}$$

where $\phi(x, y, t) = \Phi(x, y) \cos \omega t$. The wave pressure is found from the linearised form of Bernoulli's equation as

$$p = \Phi \sin \omega t$$

and the total wave force per unit span on the wall by

$$\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j} = -\int p\boldsymbol{n}\,\mathrm{d}s$$

where $\boldsymbol{n}$ is the unit outward normal to the wall, $s$ is arc-length along the wall and the integration proceeds along the wall from $(f(-1), -1)$ to $(0, 0)$. It is readily shown that

$$\boldsymbol{n}\,\mathrm{d}s = (\boldsymbol{i} - f'(y)\boldsymbol{j})\mathrm{d}y$$

and so

$$\boldsymbol{F} = -\sin \omega t \int_{-1}^{0} (\boldsymbol{i} - \boldsymbol{j}f'(y))\Phi(f(y), y)\mathrm{d}y. \tag{8}$$

It is clear that we are not going to be able to solve analytically the problem for $\Phi$ for arbitrary wall profiles $f(y)$ and hence some sort of approximate method will be required.

For the moment, we note that a shallow-water-theory solution is available when $\omega^2 < 1$ for the case of a planar outward-sloping wall $x = -\alpha y$. The appropriate solution is given on page 276 of Lamb [4] as

$$\eta = BJ_0\left(2\omega(\alpha x)^{1/2}\right) \sin \omega t$$

where $B$ is a constant. Matching this to a shallow-water standing wave at $x = \alpha$ by requiring the continuity of $\eta$ and $\mathrm{d}\eta/\mathrm{d}x$ is readily shown to give the total normal force per unit span on the wall as

$$\boldsymbol{F} = -\frac{(1+\alpha^2)^{1/2}}{\omega\alpha} \frac{J_1(2\omega\alpha) \sin \omega t \, \boldsymbol{n}}{\left\{J_0^2(2\omega\alpha) + J_1^2(2\omega\alpha)\right\}^{1/2}}. \tag{9}$$

This will be used later for comparison with our approximate solutions when the water is indeed shallow. For the moment we note that when $\alpha \ll 1$, (9) becomes

$$\boldsymbol{F} = -\left(1 + \tfrac{1}{2}\alpha^2 + O(\alpha^4)\right)\boldsymbol{n} \sin \omega t. \tag{10}$$

## 3. The perturbation solution

When the sea wall is almost vertical, a natural approach is to seek a perturbation solution in which the small parameter is a measure of the reciprocal of the slope of the wall. Thus we take the equation of the wall to be

$$x = f(y) = \alpha g(y)$$

where $|\alpha| \ll 1$ and $g$ is an $O(1)$ function with $g(0) = 0$. If $g(y) \geqslant 0$ for $-1 < y < 0$, positive values of $\alpha$ correspond to outward-sloping walls and negative values of $\alpha$ to backward-sloping walls like overhanging cliffs. Since $|\alpha|$ is assumed to be small, the boundary condition on the wall will be transferred to $x = 0$ by means of a Taylor-series expansion in $x$. This means that we now have to solve Laplace's equation for $x > 0$, $0 > y > -1$, subject to (4) at $y = -1$ for $x > 0$, (5) at $y = 0$ for $x > 0$ and a complicated boundary condition (given in (12) below) at $x = 0$ for $0 > y > -1$ and the condition (7) as $x \to \infty$.

Separation of variables using the boundary conditions at $y = 0$ and $y = -1$ leads to a solution of the form

$$\Phi = \operatorname{sech} k\left\{\cosh k(1+y) \cos(kx+\beta) + \sum_{n=1}^{\infty} A_n \cos \kappa_n(1+y) \exp(-\kappa_n x)\right\}, \tag{11}$$

where $k$ is the surface-mode wavenumber and satisfies

$$\omega^2 = k \tanh k$$

and $\kappa_n$ is the "wavenumber" of the $n$-th evanescent mode and hence satisfies

$$\omega^2 = -\kappa_n \tan \kappa_n.$$

In (11) the sech $k$ term is taken as a common factor for algebraic and numerical convenience. Essentially, it ensures that the $A_n$ do not become large in deep water ($\omega^2 \gg 1$). The solution (11) satisfies all the conditions of the problem except that of no normal flow at the wall itself. In the next section we will present a numerical technique for choosing the unknowns $\beta$ and $A_n$ to approximately satisfy (6). For the present, we note that if $\alpha = 0$, (11) satisfies the boundary condition at the wall with $\beta = 0$ and $A_n = 0$. Hence, for small $\alpha$ we postulate expansions of the form

$$\beta = \sum_{m=1}^{\infty} \beta_m \alpha^m \quad \text{and} \quad A_n = \sum_{m=1}^{\infty} A_{nm} \alpha^m.$$

If we define

$$\Phi_m = \sum_{n=1}^{\infty} A_{nm} \cos \kappa_n (1+y) \exp(-\kappa_n x)$$

and transfer the boundary condition (6) at $x = \alpha g(y)$ to $x = 0$ by means of a Taylor-series expansion in $x$, we find that this boundary condition becomes

$$\alpha\left\{-\beta_1 k \cosh k(1+y) + \frac{\partial \Phi_1}{\partial x} - g'(y) k \sinh k(1+y) - g(y) k^2 \cosh k(1+y)\right\}$$

$$+\alpha^2\left\{-\beta_2 k \cosh k(1+y) + \frac{\partial \Phi_2}{\partial x} - g'(y)\frac{\partial \Phi_1}{\partial y} + g(y)\frac{\partial^2 \Phi_1}{\partial x^2}\right\} + O(\alpha^3)$$

$$= 0 \qquad \text{at } x = 0 \quad \text{for } -1 < y < 0. \tag{12}$$

We now use the fact that the set

$$\{\cosh k(1+y);\ \cos \kappa_n(1+y),\ n = 1, 2, 3 \ldots\}, \tag{13}$$

is a complete orthogonal set of eigenfunctions on $(-1, 0)$. Hence we multiply (12) by each of these eigenfunctions, integrate from $-1$ to 0 and equate to zero the coefficient of each power of $\alpha$ which results. After some integrations by parts and use of the condition $g(0) = 0$, we find

$$\beta_1 = \frac{4k^2 \int_{-1}^{0} g(y) \sinh^2 k(1+y) \mathrm{d}y}{2k + \sinh 2k}$$

and

$$A_{n1} = \frac{-4k\kappa_n \int_{-1}^{0} g(y) \sinh k(1+y) \sin \kappa_n(1+y) \mathrm{d}y}{2\kappa_n + \sin 2\kappa_n}$$

from the $O(\alpha)$ terms. Similarly, the $O(\alpha^2)$ terms give

$$\beta_2 = \frac{-4\sum_{j=1}^{\infty} A_{j1}\kappa_j \int_{-1}^{0} g(y)\sin\kappa_n(1+y)\cosh k(1+y)\,\mathrm{d}y}{2k+\sinh 2k}$$

and

$$A_{n2} = \frac{4\sum_{j=1}^{\infty} A_{j1}\kappa_j \int_{-1}^{0} g(y)\sin\kappa_n(1+y)\sin\kappa_j(1+y)\,\mathrm{d}y}{2\kappa_n+\sin 2\kappa_n}.$$

Correct to $O(\alpha^2)$ the velocity potential on the wall is given by Taylor expansion about $x=0$ as

$$\Phi = \operatorname{sech} k\left\{\cosh k(1+y) + \alpha\sum_{n=1}^{\infty} A_{n1}\cos\kappa_n(1+y)\right.$$
$$\left. -\alpha^2\left[\tfrac{1}{2}(\beta_1+kg(y))^2\cosh k(1+y) - \sum_{n=1}^{\infty}(A_{n2}-g(y)\kappa_n A_{n1})\cos\kappa_n(1+y)\right]\right\}.$$

When this is inserted into (8) we obtain the $x$-component of the wave force per unit span on the wall correct to $O(\alpha^2)$ and the $y$-component correct to $O(\alpha^3)$. Naturally, in calculating the normal force to $O(\alpha^2)$ we only include the $O(\alpha^2)$ part of the $y$-component. For future reference, we note that, for a vertical plane wall, $\alpha=0$ and

$$\boldsymbol{F} = -k^{-1}\tanh k\,\boldsymbol{i}\sin\omega t.$$

## 4. A Galerkin solution

As remarked in Section 3, the expression (11) satisfies all the conditions of the problem except the boundary condition on the wall itself. If we rewrite it as

$$\Phi = \left(1+B_0^2\right)^{-1/2}\operatorname{sech} k\left\{(\cos kx + B_0\sin kx)\cosh k(1+y)\right.$$
$$\left. + \sum_{n=1}^{\infty} B_n\cos\kappa_n(1+y)\exp(-\kappa_n x)\right\}, \tag{14}$$

and substitute into the boundary condition (6) at the wall, we get a complicated expression which we will write as

$$B_0H_0(y) + \sum_{n=1}^{\infty} B_nH_n(y) - H(y) = 0 \qquad \text{for } -1<y<0, \tag{15}$$

after cancelling off the $(1 + B_0^2)^{-1/2}$ factor. Many methods, such as collocation, could be used to solve (15) for the $B_j$ coefficients. Here we use a Galerkin method based upon the complete set of eigenfunctions (13). Thus, we multiply (15) by cosh $k(1 + y)$ and integrate from $-1$ to 0 to give an expression which we will write as

$$\sum_{n=0}^{\infty} C_{0n} B_n = D_0. \tag{16}$$

Similarly, from the cos $\kappa_m(1 + y)$ eigenfunctions, we obtain

$$\sum_{n=0}^{\infty} C_{mn} B_n = D_m \qquad \text{for } m = 1, 2, 3, \ldots \tag{17}$$

Truncating (16) and (17) at $n = N$ gives a set of $N + 1$ linear algebraic equations for the $N + 1$ unknowns $B_0$, $B_1, \ldots B_N$. For an arbitrary wall profile $f(y)$ the various integrals occurring in these expressions would have to be evaluated numerically. Once the $B_j$ coefficients are known, we find the velocity potential from (14) and the force per unit span on the wall from (8). Again, numerical integration would be necessary for arbitrary $f(y)$.

It will be observed that this Galerkin method, although motivated by the "large-slope" solution of Section 3, makes no assumption that the slope of the wall is in fact large. However, the form of the solution is one which is appropriate to regions bounded by a flat bottom at $y = -1$ and we might expect that some problems with the method would become apparent if the slope of the wall was not large.

## 5. Results and discussion

In the two previous sections we have presented a perturbation technique and a Galerkin technique for the approximate solution of our problem. In one sense, the Galerkin method supersedes the perturbation calculation. Nevertheless, it is of interest to present the results of both in a specific case. In this section, we present results for the simple but important case of a planar outward sloping wall $x = -\alpha y$. All the integrals involved can then be evaluated analytically. As a check, the integrals were evaluated numerically using Romberg integration for a few randomly-chosen values of the parameters $\omega$ and $\alpha$. In both the perturbation and Galerkin methods, the number of terms used in the series expansions (11) and (14) was increased until the forces had converged to at least three decimal places. For the perturbation method, this typically required about 20–40 terms while for the Galerkin method it typically required 5–10 terms. As a general rule, increasing $\omega$ or $\alpha$ increased the number of terms needed for convergence.

We begin by discussing the situation in shallow water. Figure 2 shows the magnitude $F_T$ of the total normal force on the wall (the sin $\omega t$ factor being omitted) for $\omega = 0.1$ and for values of $\alpha$ ranging from 0 to 1.0. In this figure, we have shown the results of the first- and second-order perturbation methods, the Galerkin method and the shallow-water-theory solution. Numerical problems developed with the Galerkin method for values of $\alpha$ beyond unity. Using a small number of terms in the series did not give sufficiently accurate results. However as the number of terms was increased, the resulting system (16), (17) of linear equations became more and more ill-conditioned and hence the results less

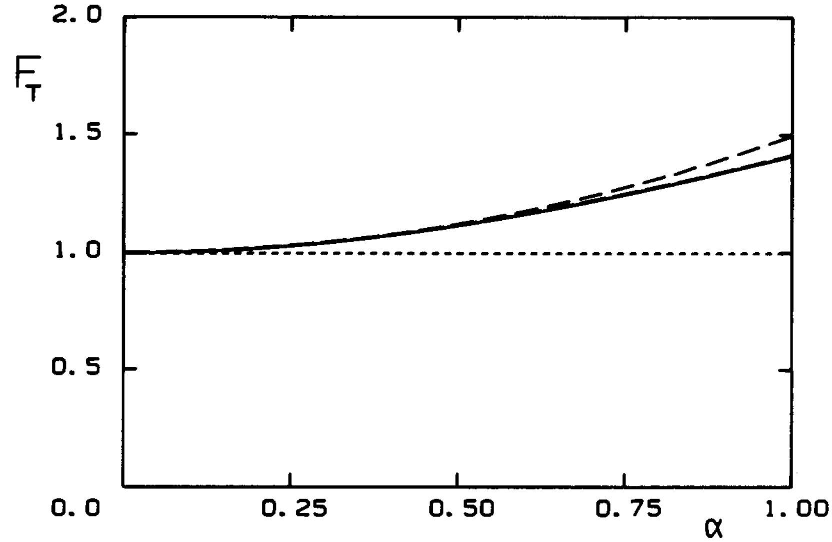


Figure 2. The non-dimensional total normal wave force $F_T$ for a planar outward-sloping sea wall $x = -\alpha y$ for $0 \leqslant \alpha \leqslant 1.0$ in the case $\omega = 0.1$ according to first (- - - - - -) and second (— — —) order perturbation theories, the Galerkin solution (———) and shallow-water theory (·—·—·). The last two are virtually indistinguishable.

and less reliable. Our VAX 11/750 uses 14 decimal places and we found that $\alpha = 1.0$ was about as far as we could go for 3 figure accuracy in $F_T$. As can be seen from Figure 2, the first-order perturbation solution gives a normal force which is virtually independent of the slope of the wall. This is consistent with (10). The second-order-perturbation, Galerkin and shallow-water-theory results all agree quite well up to about $\alpha = 0.5$. Between this value and $\alpha = 1.0$, the Galerkin and shallow-water results still agree quite closely but the second-order-perturbation results are beginning to over-estimate the force. Perhaps the most interesting aspect of Figure 2 is the agreement between the results of shallow-water theory and the (presumably numerically accurate) Galerkin method for very steeply sloping walls. This is at first sight surprising since vertical accelerations would be expected to be large in the neighbourhood of the wall, thus invalidating shallow-water theory. However, we note that shallow-water theory also gives correct predictions of certain gross features of the flow in other situations involving long waves where one would not expect vertical accelerations to be small. For example, it correctly predicts the reflection coefficient from a step discontinuity in the depth, as discussed by Bartholomeusz [1] and Tuck [8].

Turning now to the case of water of intermediate depth, we show in Figure 3 the results of the first- and second-order perturbation calculations, plus the Galerkin results, for $\omega = 1.25$ and for values of $\alpha$ ranging from 0 to 0.75, the latter being the largest value of $\alpha$ for which the Galerkin method gave reliable results. In this figure, the total normal force per unit span has been divided by its value $\tanh k/k$ for a vertical wall. This manner of presentation enables us to read off the fractional change in the force as the wall becomes less and less vertical. As can be seen, the force is only fairly weakly dependent upon the slope of the wall in this case and the three approximations merge as $\alpha \to 0$.

The corresponding results for deep water, $\omega = 2.5$, are shown in Figure 4 for values of $\alpha$ from 0 to 0.5, the latter being the largest value of $\alpha$ for which the Galerkin results were

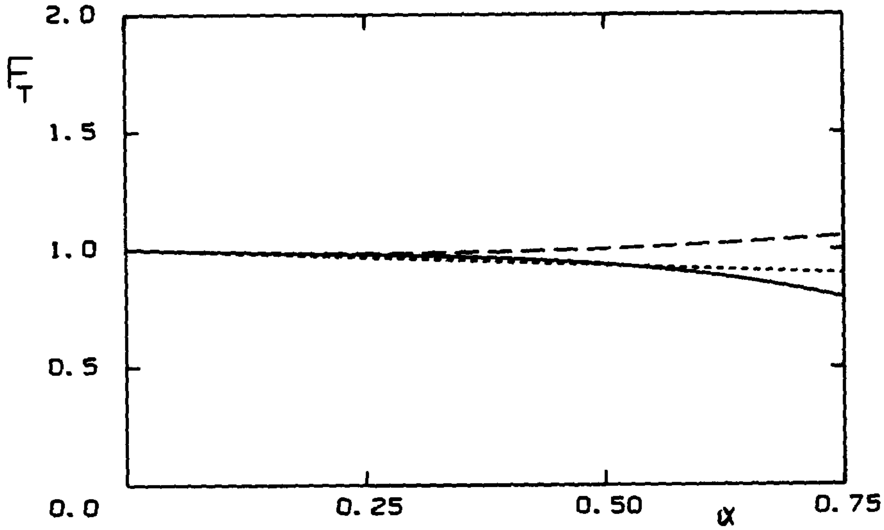


Figure 3. As for Figure 2 in the case $\omega = 1.25$ for $0 \leqslant \alpha \leqslant 0.75$. The shallow-water-theory solution is not shown and $F_T$ has been divided by the value $\tanh k/k$ ($= 0.55631$) which it takes for a vertical wall $\alpha = 0$.

reliable at this frequency. In contrast with shallow water, the total normal force now decreases as the slope $1/\alpha$ decreases and the three approximations again merge as $\alpha \to 0$. It should be noted that $\omega = 2.5$ corresponds to really quite deep water with the deep-water dispersion relation $\omega^2 = k$ being satisfied to better than 1 part in $10^5$.

From now on, we will present only the results of the Galerkin method. In Figure 5 we have plotted the total normal force per unit span for various values of $\omega$ for $0 < \alpha < 0.5$. Some of this information is presented in a different way in Figure 6 where we show the

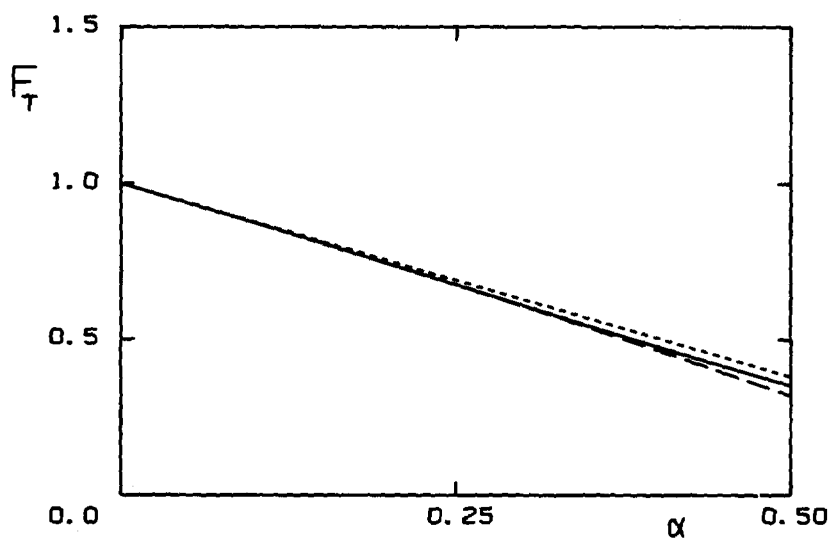


Figure 4. As for Figure 3 in the case $\omega = 2.5$ for $0 \leqslant \alpha \leqslant 0.5$. Here, $\tanh k/k = 0.16000$.

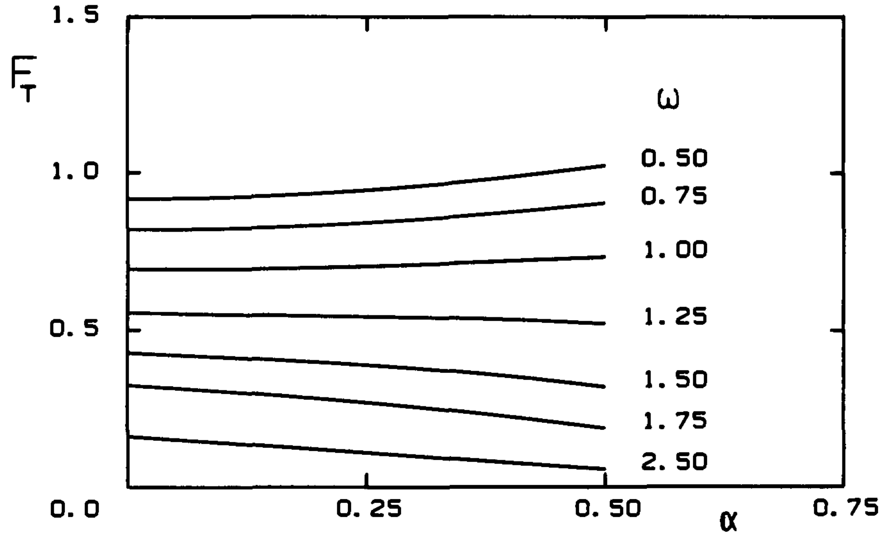


Figure 5. The total normal wave force $F_T$ as a function of $\alpha$ for several different values of $\omega$.

total normal force per unit span as a function of $\omega$ for three different values of $\alpha$; namely $\alpha = 0.0$, 0.25 and 0.5. From these figures, it is clear that in shallow water, the force increases as $\alpha$ increases while in deep water the reverse is true. In water of intermediate depth the force does not vary much as a function of $\alpha$. Indeed, a little numerical experimentation indicates that, at about $\omega = 1.14$, the force is virtually independent of $\alpha$. Perhaps the most striking feature of these results is the dramatic decrease in the wave force in deep water as the wall becomes less steep. For example, when $\alpha = 0.5$, which

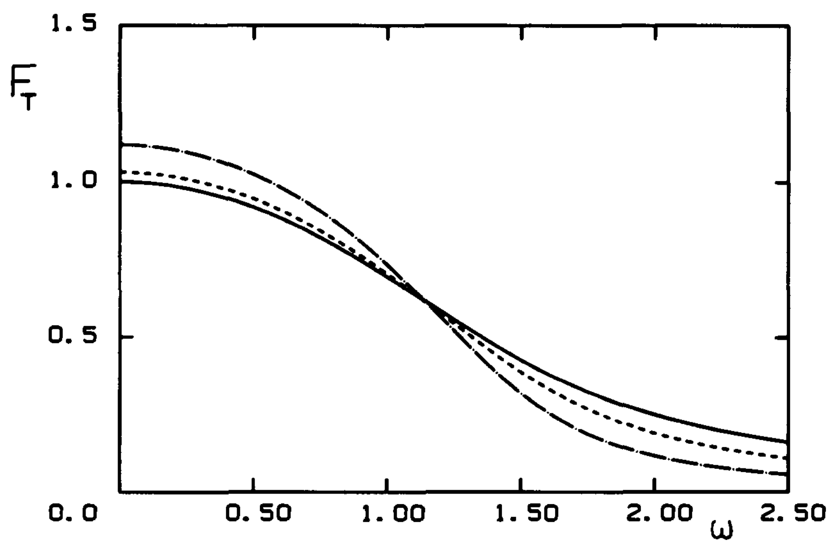


Figure 6. The total normal wave force $F_T$ as a function of $\omega$ for three different values of $\alpha$; namely $\alpha = 0.0$ (———), $\alpha = 0.25$ (- - - - - -) and $\alpha = 0.5$ (·—·—·).

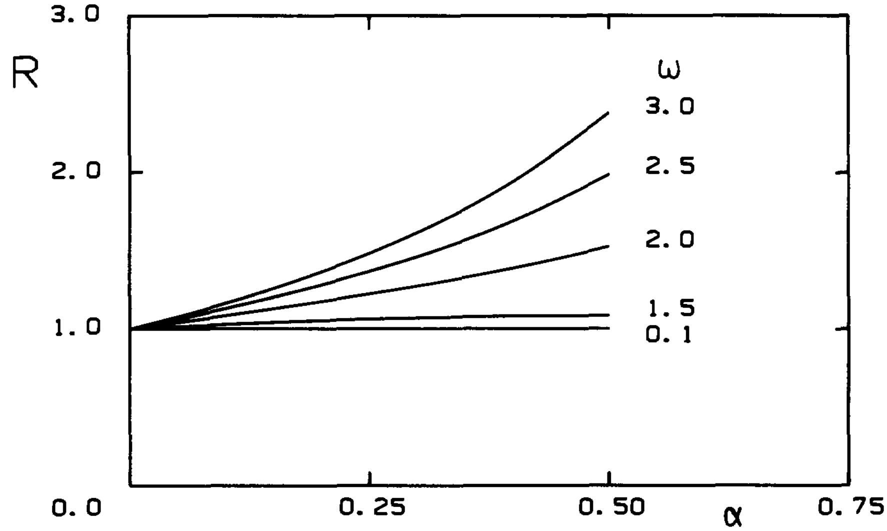


Figure 7. The ratio $R = F_S/F_T$ for $0 \leqslant \alpha \leqslant 0.5$ for various values of $\omega$. Here $F_S$ is that part of the total normal wave force $F_T$ due to the surface-wave mode only.

corresponds to a wall inclined at about 63° to the horizontal, the total normal wave force when $\omega = 2.5$ is reduced to about 35% of its value for a vertical wall.

Another point of interest is the way in which the force is influenced by the evanescent modes. To assess this, we took our converged Galerkin solution and calculated from it the total normal wave force per unit span due to the surface-wave mode only. This was then divided by the total normal wave force per unit span calculated earlier. The results are shown in Figure 7. Values greater than unity indicate that the evanescent modes act to reduce the force, i.e. the part of the force due to the evanescent modes has the sign opposite to that due to the surface wave. In shallow water, the ratio is seen to be extremely close to unity indicating that the contribution from the evanescent modes is negligible. In fact, it is clear from Figure 7 that the evanescent modes have little effect until the water gets quite deep. Even at $\omega = 1.5$ and $\alpha = 0.5$ the ratio is only 1.08. For larger values of $\omega$, the evanescent modes becomes more and more important. This means that some of the simple-minded arguments that one sometimes hears which involve only a consideration of the surface-wave component of the motion are going to be quite misleading in deep water.

In summary, we believe that our Galerkin method gives reliable results for steeply-sloping sea walls and indicates the qualitative difference in the behaviour of the wave force as a function of the wall slope between deep and shallow water. Both our perturbation and Galerkin methods are readily extended to the case of oblique incidence with only a small increase in algebraic complexity and we believe that the results of the Galerkin method would provide a useful check on the results of other methods which could be applied to this problem.

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